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# On the number of eigenvalues of the discrete one-dimensional Schrödinger operator with a complex potential

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**Abstract** We study the eigenvalues of the discrete Schrödinger operator with a complex potential. We obtain bounds on the total number of eigenvalues in the case where $V$ decays exponentially at infinity.

**Keywords** Discrete Schrödinger · Complex potential · Total number of eigenvalues

## 1 Introduction and main results

Let $\mathfrak{H} = \ell^2(\mathbb{Z}_+)$ be the Hilbert space of square summable sequences on $\mathbb{Z}_+ = \{1, 2, 3, \ldots\}$. Let $V : \mathfrak{H} \mapsto \mathfrak{H}$ be the operator of multiplication by a bounded complex-valued function on $\mathbb{Z}_+$. We study the spectral properties of the Schrödinger operator $H$, defined in $\mathfrak{H}$ by

$$(H u)_j = \sum_{|l-j|=1} u_l + V_j u_j, \quad \forall j \geq 2. \tag{1.1}$$

Additionally, we set

$$(Hu)_1 = u_2 + V_1 u_1.$$

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Note that $H$ is a bounded operator. The spectrum of the self-adjoint operator $H_0 = H - V$ coincides with the interval $[-2, 2]$ and is absolutely continuous. Let $\lambda_j$ denote the eigenvalues of the operator (1.1). We are interested in an estimate of the total number $N$ of eigenvalues $\lambda_j$ in the case where the sequence $V_j$ decays exponentially fast.

More precisely, we shall prove the following two theorems:

**Theorem 1.1** *The number $N$ of eigenvalues of $H$ in $\ell^2(\mathbb{Z}_+)$, counting algebraic multiplicities, satisfies*

$$N \leq \frac{1}{2\ln\Lambda}\left(\frac{2\Lambda^2}{\Lambda^2 - 1}\sum_{n=1}^{\infty}\Lambda^{2n}|V_n|\right)^2,$$

*for any $\Lambda > 1$.*

A similar result for a continuous operator was proved in [13] by Frank, Laptev and Safronov.

We also establish a slightly different estimate:

**Theorem 1.2** *The number $N$ of eigenvalues of $H$ in $\ell^2(\mathbb{Z}_+)$, counting algebraic multiplicities, satisfies*

$$N \leq \frac{1}{\ln\Lambda}\frac{\Lambda^2}{(\Lambda^2 - 1)}\left(\sum_{n=1}^{\infty}\Lambda^{n}|V_n|^{1/2}\right)^2,$$

*for any $\Lambda > 1$.*

Note that the right hand sides of both estimates can be finite only in the case where $V$ is an exponentially decaying potential. It turns out that $N$ might be finite even in the case when the potential decays slower. For instance, the operator $-d^2/dx^2 + V(x)$ on the half-line $[0, \infty)$ has finitely many eigenvalues if $|V| \leq C\exp(-c\sqrt{x})$ for some $C, c > 0$. This remarkable result was proved by Pavlov in [23]. It was established that the eigenvalues can not accumulate to a point of the positive half- line, which is enough to conclute that the set of all eigenvalues is finite.

On the other hand, there is another remarkable result of Pavlov (see [24]), which says that, for any $0 < p < 1/2$, there exists a complex-valued potential $V$ satisfying $|V| \leq C\exp(-c|x|^p)$ and a complex number $\theta$, such that the operator $-d^2/dx^2+V(x)$ with the boundary condition $\psi'(0) = \theta\psi(0)$ has infinitely many eigenvalues. Another interesting result was recently established by Bögli [2]. It was shown that there exists a potential for which the eigenvalues accumulate to every point on $[0, \infty)$.

## 2 Zeroes of analytic functions

The following proposition gives a useful bound on the zeroes of an analytic function in the compliment of the disc of radius $R > 0$.

**Proposition 2.1** *Let $0 < R < 1$. Let $a(\cdot)$ be an analytic function in $\{k\colon\ |k| > R\}$. Assume that $a(\cdot)$ is continuous up to the boundary and satisfies*

$$a(k) = 1 + O(|k|^{-1}) \quad \text{as } |k| \to \infty \text{ in } \{k\colon\ |k| > R\}. \tag{2.1}$$

*Assume also that for some $A \geq 1$,*

$$|a(k)| \leq A, \quad \text{if } |k| = R\,. \tag{2.2}$$

*Then the zeroes $k_j$ of $a(\cdot)$ in $\{k\colon\ |k| > R\}$, repeated according to their multiplicities, satisfy*

$$\prod_j \left(\frac{|k_j|}{R}\right) \leq A. \tag{2.3}$$

*Proof* We introduce the Blaschke product

$$B(k) = \prod_j \frac{k - k_j}{R - R^{-1}\overline{k_j}k}\,.$$

Clearly, $a(k)/B(k)$ is an analytic and non-zero in $\{k\colon\ |k| > R\}$. Consequently, $\log(a(k)/B(k))$ exists and is analytic in $\{k\colon\ |k| > R\}$. Let $C_R$ denote the circle $\{k \in \mathbb{C}\colon\ |k| = R\}$, traversed counterclockwise.

Then, according to the residue calculus,

$$\int_{C_R} \log \frac{a(k)}{B(k)} \frac{dk}{k} = 2\pi i \lim_{k\to\infty} \log \frac{a(k)}{B(k)} = 2\pi i \sum_j \log \frac{\bar{k}_j}{-R}\,,$$

and therefore

$$\int_{-\pi}^{\pi} \log \frac{a(Re^{i\varphi})}{B(Re^{i\varphi})}\, d\varphi = 2\pi \sum_j \log \frac{\bar{k}_j}{-R}\,. \tag{2.4}$$

We note that $|B(Re^{i\varphi})| = 1$ if $\varphi \in \mathbb{R}$ and, therefore,

$$\operatorname{Re} \int_{-\pi}^{\pi} \log \frac{a(Re^{i\varphi})}{B(Re^{i\varphi})}\, d\varphi = \int_{-\pi}^{\pi} \ln \left|\frac{a(Re^{i\varphi})}{B(Re^{i\varphi})}\right| d\varphi = \int_{-\pi}^{\pi} \ln |a(Re^{i\varphi})|\, d\varphi. \tag{2.5}$$

On the other hand,

$$\operatorname{Re} \sum_j \log \frac{\bar{k}_j}{-R} = \sum_j \ln \frac{|k_j|}{R}. \tag{2.6}$$

We conclude from (2.4), (2.5) and (2.6) that

$$\int_{-\pi}^{\pi} \ln |a(Re^{i\varphi})|\, d\varphi = 2\pi \sum_j \ln \frac{|k_j|}{R}. \tag{2.7}$$

Finally, by (2.2),

$$\int_{-\pi}^{\pi} \ln |a(Re^{i\varphi})| \, d\varphi \leq 2\pi \ln A, . \tag{2.8}$$

Inequality (2.3) now follows from (2.7) and (2.8). □

**Corollary 2.2** *Let* $0 < R < 1$. *Let* $a(\cdot)$ *be an analytic function in* $\{k\colon |k| > R\}$ *satisfying* (2.1). *Assume that, for any* $R' > R$ *sufficiently close to* $R$, *condition* (2.2) *holds with* $R$ *replaced by* $R'$. *Then the number*

$$\mathcal{N} := \#\{j\colon \ |k_j| \geq 1\}$$

*of zeroes* $k_j$ *of* $a(\cdot)$ *in* $\{k\colon \ |k| \geq 1\}$, *repeated according to their multiplicities, satisfies*

$$\mathcal{N} \leq \frac{\ln A}{\ln 1/R} \, .$$

*Proof* We apply Proposition 2.1 for every $R' > R$ sufficiently close to $R$ and obtain

$$\sum_j \left(\ln |k_j| - \ln R'\right)_+ \leq \ln A$$

Clearly, we have

$$\sum_j \left(\ln |k_j| - \ln R'\right)_+ \geq |\ln R'| \cdot \# \left\{j\colon \ |k_j| \geq 1\right\} \, .$$

Consequently,

$$|\ln R'| \cdot \mathcal{N} \leq \ln A \, .$$

The corollary follows by passing to the limit $R' \to R$. □

## 3 Classes of compact operators and determinants

Let $1 \leq p < \infty$. We say that a compact operator $T$ belongs to the Schatten class $\mathfrak{S}_p$ if its singular values $s_j(T)$ satisfy

$$\|T\|_{\mathfrak{S}_p}^p := \sum_j s_j^p(T) < \infty.$$

The functional $\| \cdot \|_{\mathfrak{S}_p}$ is the norm on $\mathfrak{S}_p$.

Let $K \in \mathfrak{S}_n$ with $n \in \mathbb{N}$. Let $\lambda_j(K)$ denote the eigenvalues of $K$, repeated according to algebraic multiplicities. The $n$-th order regularized determinant $\det_n(1+K)$ is defined by

$$\det{}_n(1+K) := \prod_j \left( \big(1+\lambda_j(K)\big) \exp\left( \sum_{m=1}^{n-1} \frac{(-1)^m}{m} \lambda_j(K)^m \right) \right).$$

The following property is well-known, but we include a proof for the sake of completeness.

**Lemma 3.1** *Let $n \in \mathbb{N}$ and let $K \in \mathfrak{S}_n$. Then*

$$\ln|\det{}_n(1+K)| \leq \Gamma_n \|K\|_{\mathfrak{S}_n}^n ,$$

*where $\Gamma_n$ is a positive constant independent of $K$. In particular,*

$$\Gamma_1 = 1 \quad \textit{and} \quad \Gamma_2 = 1/2 . \tag{3.1}$$

*Proof* To prove the lemma, let $f(z) := (1+z)\exp\left(\sum_{m=1}^{n-1} \frac{(-1)^m}{m} z^m\right)$. Then $\ln|f(z)|$ can be bounded by a constant times $|z|^n$ for small $|z|$ and by a constant times $|z|^{n-1}$ for large $|z|$. Thus, $\ln|f(z)| \leq \Gamma_n |z|^n$, and so

$$\ln|\det{}_n(1+K)| \leq \Gamma_n \sum_j |\lambda_j(K)|^n$$

By Weyl's inequality [26, Thm. 1.15], the sum on the right side does not exceed $\|K\|_{\mathfrak{S}_n}^n$. A simple computation shows that for $n = 1$ and $n = 2$ one can take $\Gamma_1 = 1$ and $\Gamma_2 = 1/2$, respectively (see [27]). □

We now recall the Birman–Schwinger principle. We state it in the case, where $H_0$ is a bounded self-adjoint operator and $V = G^* G_0$. We will assume that $G_0$ and $G$ are compact operators. Now, set

$$H = H_0 + V.$$

The Birman–Schwinger principle states that $z \in \rho(H_0)$ is an eigenvalue of $H$ if and only if $-1$ is an eigenvalue of the Birman–Schwinger operator $G_0(H_0 - z)^{-1} G^*$. Moreover, the corresponding geometric multiplicities coincide.

The following lemma says that even the algebraic multiplicities of eigenvalues of $H$ can be characterizes in terms of a quantity related to the Birman–Schwinger operator.

**Lemma 3.2** *Let $n \in \mathbb{N}$. Assume that $G_0(H_0-\zeta)^{-1}G^* \in \mathfrak{S}_n$ for all $\zeta \in \rho(H_0)$. Then the function $\zeta \mapsto \det_n(1+G_0(H_0-\zeta)^{-1}G^*)$ is analytic in $\rho(H_0)$. A point $z \in \rho(H_0)$ is an eigenvalue of $H$ if and only if $\det_n(1+G_0(H_0-z)^{-1}G^*) = 0$. Moreover, the order of the zero coincides with the algebraic multiplicity of the corresponding eigenvalue.*

Analyticity of the function $\zeta \mapsto \det_n(1 + G_0(H_0 - \zeta)^{-1}G^*)$ is well-known (see, e.g., [27]), as well as the result about the algebraic multiplicity in the case $n = 1$. The result for the general $n$ is essentially due to [18]; you may also refer to [11] for an extension of the proof to the present setting.

## 4 Resolvent bounds

In this section we collect trace ideal bounds for the Birman–Schwinger operator

$$K(k) = \sqrt{V}(H_0 - z)^{-1}\sqrt{|V|}, \quad z = k + k^{-1}, \quad |k| \geq 1\,. \tag{4.1}$$

We use the notation $\sqrt{V(x)} = V(x)/\sqrt{|V(x)|}$ if $V(x) \neq 0$ and $\sqrt{V(x)} = 0$ if $V(x) = 0$.

We remind the reader that $\mathfrak{H} = \ell^2(\mathbb{Z}_+)$, and $H_0$ in (4.1) denotes the free Jacobi operator on $\mathbb{Z}_+$. From the explicit expression of its matrix it is easy to see that, if $V$ has a compact support, then $K(k)$ admits an analytic continuation to $\mathbb{C} \setminus \{0\}$. The following proposition gives a bound on the Hilbert–Schmidt norm.

**Proposition 4.1** *For any $k \in \mathbb{C}\backslash\{0\}$ with $|k| < 1$,*

$$\|K(k)\|_{\mathfrak{S}_2} \leq \frac{2}{1-|k|^2}\sum_{n=1}^{\infty}|k|^{-2n}|V_n|\,,$$

*Proof* The matrix of $(H_0 - z)^{-1}$ is given by

$$g_k(n,m) = \frac{k}{k^2-1}\left(k^{-|n-m|} - k^{-(n+m)}\right)\,,$$

which satisfies

$$|g_k(n,m)| \leq \frac{2}{1-|k|^2}|k|^{-(n+m)}\,.$$

Combining this bound with the identity

$$\|K(k)\|_{\mathfrak{S}_2}^2 = \sum_1^{\infty}\sum_1^{\infty}|V_n||g_k(n,m)|^2|V_m|$$

we obtain the claimed bound.

□

**Proposition 4.2** *For any $k \in \mathbb{C}\backslash\{0\}$ with $|k| < 1$,*

$$\|K(k)\|_{\mathfrak{S}_1} \leq \frac{2}{1-|k|^2}\left(\sum_{n=1}^{\infty}|k|^{-n}|V_n|^{1/2}\right)^2,$$

*Proof* The matrix of $(H_0 - z)^{-1}$ is defined by

$$g_k(n,m) = \frac{k}{k^2-1}\left(k^{-|n-m|} - k^{-(n+m)}\right),$$

which satisfies

$$|g_k(n,m)| \le \frac{2}{1-|k|^2}|k|^{-(n+m)}.$$

Combining this bound with the identity

$$\|K(k)\|_{\mathfrak{S}_1} \le \sum_1^\infty \sum_1^\infty |V_n|^{1/2}|g_k(n,m)||V_m|^{1/2}$$

we obtain the claimed bound. □

## 5 Proof of Theorem 1.1

In this section we prove Theorem 1.1. Let us assume that $V$ has compact support. The bound in this case implies the bound in the general case by a simple continuity argument.

As discussed in Sect. 4, the Birman–Schwinger operators $K(k)$ from (4.1) extends analytically to $\mathbb{C}\setminus\{0\}$. The same proof shows that they are not only analytic with respect to the norm of bounded operators, but even with respect to the norm in $\mathfrak{S}_2$.

We will apply Corollary 2.2 to the function

$$a(k) := \det{}_2(1 + K(k))$$

with $\Lambda = 1/R$. Since $K(k)$ is analytic with values in $\mathfrak{S}_2$, the function $a$ is analytic. It is easy to see that assumption (2.1) is valid. Moreover, combining them with Lemma 3.1, we see that assumption (2.2) holds with

$$\ln A = \frac{1}{2}\left(\frac{2\Lambda^2}{\Lambda^2-1}\sum_{n=1}^\infty \Lambda^{2n}|V_n|\right)^2.$$

Thus, Corollary 2.2 implies that

$$\#\{j:\quad \operatorname{Im} k_j \ge 0\} \le \frac{1}{2\ln\Lambda}\left(\frac{2\Lambda^2}{\Lambda^2-1}\sum_{n=1}^\infty \Lambda^{2n}|V_n|\right)^2.$$

It remains to use Lemma 3.2, which says that the $k_j + k_j^{-1}$, with $|k_j| > 1$, coincide with the eigenvalues of $H$, counting algebraic multiplicities. This proves Theorem 1.1.

## 6 Proof of Theorem 1.2

In this section we prove Theorem 1.2. Let us assume again that $V$ has compact support.

As discussed in Sect. 4, the Birman–Schwinger operators $K(k)$ from (4.1) extend analytically to $\mathbb{C}\setminus\{0\}$. The same proof shows that they are not only analytic with respect to the norm of bounded operators, but even with respect to the norm in $\mathfrak{S}_1$.

We apply Corollary 2.2 to the function

$$a(k) := \det{}_1(1 + K(k)) = \det(1 + K(k))$$

with $\Lambda = 1/R$. Since $K(k)$ is analytic with values in $\mathfrak{S}_1$, the function $a$ is analytic. Assumption (2.2) holds with

$$\ln A = \left( \frac{2\Lambda^2}{\Lambda^2 - 1} \sum_{n=1}^{\infty} \Lambda^n |V_n|^{1/2} \right)^2 .$$

Thus, Corollary 2.2 implies that

$$\#\{j: \quad \operatorname{Im} k_j \geq 0\} \leq \frac{1}{\ln \Lambda} \left( \frac{2\Lambda^2}{\Lambda^2 - 1} \sum_{n=1}^{\infty} \Lambda^n |V_n|^{1/2} \right)^2 .$$

It remains to use Lemma 3.2, which says that the $k_j + k_j^{-1}$, with $|k_j| > 1$, coincide with the eigenvalues of $H$, counting algebraic multiplicities. This proves Theorem 1.2. □

Most of the papers listed below contain results on the eigenvalues of non-selfadjoint operators. More specifically, those are the articles [1–17, 19–25, 28, 29]. The remaining references were needed for technical reasons.



## References

1. Abramov, A.A., Aslanyan, A., Davies, E.B.: Bounds on complex eigenvalues and resonances. J. Phys. A **34**, 57–72 (2001)
2. Bögli, S.: Schrödinger operator with non-zero accumulation points of complex Eigenvalues (2016). arxiv:1605.09356 (**Preprint**)
3. Borichev, A., Golinskii, L., Kupin, S.: A Blaschke-type condition and its application to complex Jacobi matrices. Bull. Lond. Math. Soc. **41**, 117–123 (2009)
4. Davies, E.B.: Non-self-adjoint differential operators. Bull. Lond. Math. Soc. **34**(5), 513–532 (2002)
5. Davies, E.B., Nath, J.: Schrödinger operators with slowly decaying potentials. J. Comput. Appl. Math. **148**, 1–28 (2002)
6. Demuth, M., Katriel, G.: Eigenvalue inequalities in terms of Schatten norm bounds on differences of semigroups, and application to Schrödinger operators. Ann. Henri Poincaré **9**(4), 817–834 (2008)

7. Demuth, M., Hansmann, M., Katriel, G.: On the discrete spectrum of non-selfadjoint operators. J. Funct. Anal. **257**(9), 2742–2759 (2009)
8. Demuth, M., Hansmann, M., Katriel, G.: Eigen values of non-self adjoint operators: a comparison of two approaches. In: Mathematical Physics, Spectral Theory and Stochastic Analysis, Springer, pp. 107–163 (2013)
9. Enblom, A.: Estimates for eigenvalues of Schrödinger operators with complex-valued potentials (2015). arxiv:1503.06337 (**Preprint**)
10. Frank, R.L.: Eigenvalue bounds for Schrödinger operators with complex potentials. Bull. Lond. Math. Soc. **43**(4), 745–750 (2011)
11. Frank, R.L.: Eigenvalue bounds for Schrödinger operators with complex potentials. III (2015). arxiv:1510.03411v1 (**Preprint**)
12. Frank, R.L., Laptev, A., Lieb, E.H., Seiringer, R.: Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials. Lett. Math. Phys. **77**, 309–316 (2006)
13. Frank, R.L., Laptev, A., Safronov, O.: On the number of eigenvalues of Schrödinger operators with complex potentials. to appear
14. Frank, R.L., Laptev, A., Seiringer, R.: A sharp bound on eigenvalues of Schrödinger operators on the half-line with complex-valued potentials. Spectral theory and analysis, pp. 39–44, Oper. Theory Adv. Appl. **214**, Birkhäuser/Springer Basel AG, Basel (2011)
15. Frank, R.L., Sabin, J.: Restriction theorems for orthonormal functions, Strichartz inequalities and uniform Sobolev estimates (2014). arxiv:1404.2817 (**Preprint**)
16. Frank, R.L., Simon, B.: Eigenvalue bounds for Schrödinger operators with complex potentials. II. J. Spectr. Theory (**to appear**)
17. Laptev, A., Safronov, O.: Eigenvalue estimates for Schrödinger operators with complex potentials. Comm. Math. Phys. **292**, 29–54 (2009)
18. Latushkin, Y., Sukhtayev, A.: The algebraic multiplicity of eigenvalues and the Evans function revisited. Math. Model. Nat. Phenom. **5**(4), 269–292 (2010)
19. Martirosjan, R.M.: On the spectrum of the non-selfadjoint operator $\Delta u + cu$ in three dimensional space. (Russian) Izv. Akad. Nauk Armyan. SSR. Ser. Fiz.-Mat. Nauk **10**(1), 85–111 (1957)
20. Martirosjan, R.M.: On the spectrum of various perturbations of the Laplace operator in spaces of three or more dimensions. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **24**, 897–920 (1960)
21. Murtazin, K.K.: Spectrum of the nonself-adjoint Schrödinger operator in unbounded regions. (Russian) Mat. Zametki **9**, 19–26. English translation: Math. Notes **9**(1971), 12–16 (1971)
22. Naĭmark, M.A.: Investigation of the spectrum and the expansion in eigenfunctions of a nonselfadjoint operator of the second order on a semi-axis. (Russian) Trudy Moskov. Mat. Obšč. **3**, 181–270 (1954)
23. Pavlov, B.S.: On a non-selfadjoint Schrödinger operator. (Russian) 1966 Probl. Math. Phys., No. 1, Spectral Theory and Wave Processes (Russian) pp. 102–132 Izdat. Leningrad. Univ., Leningrad
24. Pavlov, B.S.: On a non-selfadjoint Schrödinger operator. II. (Russian) 1967 Problems of Mathematical Physics, No. 2, Spectral Theory, Diffraction Problems (Russian) pp. 133–157 Izdat. Leningrad. Univ., Leningrad
25. Safronov, O.: On a sum rule for Schrödinger operators with complex potentials. Proc. Am. Math. Soc. **138**(6), 2107–2112 (2010)
26. Simon, B.: Trace Ideals and Their Applications, 2nd edn. Amer. Math. Soc, Providence (2005)
27. Simon, B.: Notes on infinite determinants of Hilbert space operators. Adv. Math. **24**(3), 244–273 (1977)
28. Stepin, S.A.: Complex potentials: bound states, quantum dynamics and wave operators. Semigroups of operators - theory and applications, 287–297, Springer Proc. Math. Stat. **113**, Springer, Cham (2015)
29. Stepin, S.A.: Estimate for the number of eigenvalues of the nonselfadjoint Schrödinger operator. (Russian) Dokl. Akad. Nauk **455**(4), 394–397; translation in Dokl. Math. **89**(2), 202–205 (2014)